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Limit theorems for probabilities of large deviations of a Galton–Watson process

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Abstract — We prove local and integral limit theorems for large deviations of Cramer type for a critical Galton–Watson branching process under the assumption that the radius of convergence of the generating function of the progeny is strictly greater than one. The proof is based on a modified Cramer approach which consists of construction of an auxiliary non-homogeneous in time branching process.

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1. INTRODUCTION

Let Z_n stand for a Galton–Watson process beginning with a single particle of zero generation. We set

$$p_k = \mathbf{P}(Z_1 = k), \qquad k = 0, 1, \dots,$$

$$f(s) = \sum_{k=0}^{\infty} p_k s^k,$$

thus, $f(s)$ is the generating function of the progeny of an individual. Let $f_k(s)$ denote the kth iteration of the function $f(s)$. We set $B = f''(1)$, $C = f'''(1)$. Let R stand for the convergence radius of the function $f(s)$.

For brevity, we set

$$P_n(u) = \mathbf{P}\left(\frac{2Z_n}{Bn} > u \mid Z_n > 0\right).$$

It is well known (see, e.g. [1], p. 39) that if $0 < B < \infty$, then for any fixed u

$$\lim_{n\to\infty} P_n(u) = e^{-u}. \tag{1}$$

The first estimate of convergence rate in (1) was obtained in [2] under the condition $C < \infty$, namely,

$$\Delta_n = \sup_u |P_n(u) - e^{-u}| = O\left(\frac{\ln^2 n}{n}\right). \tag{2}$$

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We observe that the constants in this estimate have a more complicated relationship to the distribution of Z_1 as compared to the classical Berry–Esseen estimate.

If we set $u_n = \ln n - (2+\varepsilon)\ln\ln n$, where $\varepsilon > 0$, then by (2)

$$\lim_{n\to\infty} \sup_{u\le u_n} e^u P_n(u) = 1. \tag{3}$$

If $u > \ln n - (2-\varepsilon)\ln\ln n$, then (2) yields nothing more than the upper estimate

$$P_n(u) = O\left(\frac{\ln^2 n}{n}\right),$$

which does not depend on u. Thus, estimate (2) for large u contains not so much information about the magnitude of the ratio $P_n(u)/e^{-u}$. As in the scheme of summation of independent random variables, the problem to estimate $P_n(u)$ for large u is much easier to solve under the condition $R > 1$.

Under this assumption, in [3] the inequality

$$\mathbf{P}(Z_n \ge k) < (1+y_0)\left(1 + \frac{1}{1/y_0 + B_0 n/2}\right)^{-k}, \tag{4}$$

is obtained, where y_0 is an arbitrary number from 0 to $R-1$, $B_0 = f''(1+y_0)$.

It is easy to see that $1/y_0 + B_0 n/2$ attains its minimum if y_0 solves the equation $f'''(1+y)y^2 = 2/n$, that is, $y_0 = O(1/\sqrt{n})$. Hence,

$$\min_{y_0}(1/y_0 + B_0 n/2) = \frac{Bn}{2} + O(\sqrt{n}).$$

Thus,

$$\mathbf{P}(Z_n \ge k) < (1+\varepsilon_n)\exp\left(-\frac{2k}{Bn}(1-\eta_n)\right), \tag{5}$$

where $\varepsilon_n > 0$, $\eta_n > 0$, and $\varepsilon_n = O(1/\sqrt{n})$, $\eta_n = O(1/\sqrt{n})$.

Inequality (4) is close to the Bernstein and Petrov inequalities (see, e.g. [4], Chapter 3, Section 5).

On the other hand, from (3) it follows that

$$\mathbf{P}(Z_n \ge k) < \frac{2(1+\varepsilon_n)}{Bn}\exp\left(-\frac{2k}{Bn}\right), \qquad \varepsilon_n \to 0, \tag{6}$$

for $k < Bn(\ln n - (2+\varepsilon)\ln\ln n)/2$, because [2]

$$Q_n = \mathbf{P}(Z_n > 0) = \frac{2}{Bn} + \left(\frac{4C}{3B^2} - \frac{2}{B}\right)\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right). \tag{7}$$

We see that (5) is of less accuracy than (6) in the domain where the latter is valid, because (5) lacks the factor $2/(Bn)$.

Under the same assumption $R > 1$, in [5] it is shown that (3) remains true if $u_n = o(n/\ln n \ln_{(N)} n)$, where $\ln_{(N)} n$ is the Nth iteration of the logarithm and $N \ge 2$. In that paper, a local limit theorem is also proved for those k which correspond to the domain $0 < u < u_n$ under the additional assumption that $\gcd\{k\colon\, p_k > 0\} = 1$. More exactly,

$$\mathbf{P}(Z_n = k) = \frac{4}{B^2 n^2} \exp\left(-\frac{2k}{Bn}\right)(1 + o(1)) \tag{8}$$

uniformly in $k = o(n^2 / \ln n \ln_{(N)} n)$.

Starting from the analogy to the scheme of summation of independent random variables, we hypothesise that for $R > 1$ there exists a domain of values of u where

$$P_n(u) = e^{-u}\Omega_n(u)(1 + o(1)), \tag{9}$$

here $\Omega_n(u)$ is an explicitly calculated correction factor, that is, an analogue of the well-known Cramer theorem is true [4]. The results obtained in the present paper are evidence in favour of this hypothesis.

Theorem 1. *Let $R > 1$, $k/n \to \infty$, $k = o(n^2)$. Then as $n \to \infty$*

$$\mathbf{P}(Z_n = k) = \frac{4d}{B^2 n^2} \exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}\left(1 + O\left(\frac{k}{n^2} + \frac{\ln n}{n}\right)\right), \tag{10}$$

for k divisible by $d = \gcd\{k\colon\, p_k > 0\}$, where $\gamma = 1 - 2C/(3B^2)$.

On the base of the local limit theorem, we arrive at the integral theorem on large deviations.

Theorem 2. *Under the hypotheses of Theorem* 1*,*

$$\mathbf{P}(Z_n \geq k) = \frac{2}{Bn} \exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}\left(1 + O\left(\frac{k}{n^2} + \frac{\ln^2 n}{n}\right)\right). \tag{11}$$

From this theorem we derive the exact boundaries where convergence to the exponential law takes place.

Corollary 1. *If $u_n = o(n/\ln n)$, then relation* (3) *is true.*

It is not difficult to see that these boundaries cannot be enlarged without additional constraints imposed on the process Z_n. If we set, say, $u_n = n/\ln n$, then by (11) and (7)

$$\lim_{n\to\infty} e^{u_n} P_n(u_n) = \exp\left(-\frac{2\gamma}{B}\right) \neq 1$$

for $\gamma \neq 0$. In the case of $\gamma = 0$, convergence to the exponential law takes place for all $u = o(n)$.

From (7) and (11) it follows that the equality

$$P_n(u) = e^{-u} \exp\left(-\frac{\gamma}{n} u \ln u\right)(1 + o(1)) \tag{12}$$

is true for $u = o(n)$, that is, $\Omega_n(u)$ in (9) satisfies the equality

$$\Omega_n(u) = \exp\left(-\frac{\gamma}{n} u \ln u\right).$$

We observe that, in contrast to the classical Cramer theorem, $\ln \Omega_n(u)$ depends on the second and third moments of the initial distribution only. This is likely due to the fact that the domain where the correction factor is of importance is quite narrow.

Our way to derive relations (10) and (11) differs much from that used in [5]. The proof of Theorem 1 is based on a modified Cramer method (see, e.g. [4], Chapter 8, Section 2), which consists of the following.

Let X be a random variable that takes non-negative integer values only. We assume that the radius of convergence of the function $\rho(s) = \mathbf{E} s^X$ is strictly greater that one. Then for any r such that $\rho(r) < \infty$ we set $\rho_r(s) = \rho(rs)/\rho(r)$.

The random variable $X(r)$ such that $\mathbf{E} s^{X(r)} = \rho_r(s)$ is referred to as the Cramer transform of the random variable X. With the use of Cramer transforms of the random variable Z_1, we construct a non-homogeneous in time Galton–Watson process Y_k, $k = 1, \dots, n$, such that the distribution of the progeny of an individual in the $(k-1)$th generation is defined by the generating function $f(r_{k-1}s)/f(r_{k-1})$, where the parameters r_k of the Cramer transform are calculated by the recurrence relation $r_k = f(r_{k-1})$. Then the distributions of the initial process and the auxiliary one are related as follows:

$$\mathbf{P}(Z_n = k) = f_n(r_0) r_0^{-k} \mathbf{P}(Y_n = k), \qquad \mathbf{P}(Z_n \geq k) = f_n(r_0) \mathbf{E}\{r_0^{-Y_n}; Y_n \geq k\}.$$

The parameter r_0 is chosen so that the large deviations of the initial process become normal ones for the auxiliary process.

Non-homogeneous in time branching processes are studied in [6], where conditions of convergence to the exponential law are obtained, as well as an estimate of the convergence rate, which coincides with (2) for a critical Galton–Watson process.

In the classical Cramer theorem, the asymptotic behaviour of the mathematical expectation which connects the distributions of the initial and auxiliary sums is found with the use of the Berry–Esseen inequality. In the case of branching processes, this method does not allow us to find the asymptotic formula for $\mathbf{E}\{r_0^{-Y_n}; Y_n \geq k\}$ because the Berry–Esseen type estimate for the auxiliary process becomes too rough.

If $d > 1$, we may reduce the case to aperiodic one. We consider the process Z_n^* constructed by the generating function

$$g(s) = (f(s^{1/d}))^d = \sum_{i=0}^{\infty} p_i^* s^i.$$

It is obvious that the convergence radius of $g(s)$ is also greater than one and

$$d^* = \gcd\{k \colon p_k^* > 0\} = 1.$$

Besides, $B^* = g''(1) = B/d$, $C^* = g'''(1) = C/d^2$. Therefore, $\gamma^* = \gamma$.

It is not difficult to see that for any $n \geq 1$

$$g_n(s) = (f_n(s^{1/d}))^d.$$

This equality means that the process Z_n^* admits the representation

$$Z_n^* = \frac{1}{d}(Z_n^{(1)} + \ldots + Z_n^{(d)}), \tag{13}$$

where $Z_n^{(i)}, i = 1, \ldots, d$, are independent random variables distributed as Z_n. This representation yields the inequality

$$\mathbf{P}(Z_n^* = k) = \mathbf{P}(Z_n^{(1)} + \ldots + Z_n^{(d)} = kd) \geq d(1 - Q_n)^{d-1}\mathbf{P}(Z_n = kd). \tag{14}$$

On the other hand [7],

$$\mathbf{P}(Z_n = kd) \geq \mathbf{P}(Z_{n-1}^* = k) \sum_{l=1}^{\infty} l p_{ld} \mathbf{P}^{l-1}(Z_n^* = 0). \tag{15}$$

It is not difficult to see that

$$\sum_{l=1}^{\infty} l p_{ld} \mathbf{P}^{l-1}(Z_n^* = 0) = (f(s^{1/d}))'|_{s=1} + O(\mathbf{P}(Z_n^* > 0)) = \frac{1}{d} + O(n^{-1}).$$

From (14), (15), and the last relation it follows that

$$\mathbf{P}(Z_n = kd) \leq \frac{1}{d}\, \mathbf{P}(Z_n^* = k)(1 + O(n^{-1})), \tag{16}$$

$$\mathbf{P}(Z_n = kd) \geq \frac{1}{d}\, \mathbf{P}(Z_{n-1}^* = k)(1 + O(n^{-1})) \tag{17}$$

uniformly in all k.

If we assume that the theorem is true for the aperiodic case, then, applying it to the process Z_n^*, we obtain

$$\mathbf{P}(Z_n^* = k) = \frac{4d^2}{B^2 n^2} \exp\left\{-\frac{2kd}{Bn} - \frac{2}{B}\gamma \frac{kd}{n^2} \ln\left(\frac{k}{n}\right)\right\}\left(1 + O\left(\frac{k}{n^2} + \frac{\ln n}{n}\right)\right).$$

Hence it follows that

$$\mathbf{P}(Z_{n-1}^* = k) = \mathbf{P}(Z_n^* = k)\left(1 + O\left(\frac{k}{n^2} + \frac{\ln n}{n}\right)\right)$$

for $k = o(n^2)$. The two last relations and inequalities (16), (17) imply the equality

$$\mathbf{P}(Z_n = kd) = \frac{4d}{B^2 n^2} \exp\left\{-\frac{2kd}{Bn} - \frac{2}{B}\gamma \frac{kd}{n^2} \ln\left(\frac{k}{n}\right)\right\}\left(1 + O\left(\frac{k}{n^2} + \frac{\ln n}{n}\right)\right),$$

that is, it is proved that validity of (10) in the aperiodic case implies validity of (10) for an arbitrary $f(s)$. Similarly it is proved that (11) remains true for $d > 1$ as well. Thus, it suffices to prove Theorems 1 and 2 under the condition $d = 1$. It is necessary to note that such a reduction to the aperiodic case was used in [7] while proving a local limit theorem. But they derived a similar to (14) estimate without use of representation (13), which contributed to the difficulties they met.

2. AUXILIARY RESULTS

Lemma 1. *Let* $0 < y_0 < R - 1$, *and let the sequence* y_j *be defined by the equation*

$$y_{j+1} = f^{-1}(1 + y_j) - 1.$$

Then

$$y_j = \frac{y_0}{1 + Bjy_0/2} + \left(\frac{B}{2} - \frac{C}{3B}\right)\frac{y_0^2}{(1 + Bjy_0/2)^2}\ln\left(1 + \frac{Bjy_0}{2}\right) + O\left(\frac{y_0^2}{(1 + Bjy_0/2)^2}\right). \tag{18}$$

This lemma, as well as its proof, is very similar to Theorem 3 in [2].

Proof. From the definition of y_j it follows that

$$y_j = f(1 + y_{j+1}) - 1.$$

It is not difficult to see that $f(1 + y) \geq 1 + y$ for $y \geq 0$. Therefore, the sequence y_j decreases. Hence the existence of a limit of y_j as $j \to \infty$ follows. It is obvious that this limit has to satisfy the equation $y = f(1 + y) - 1$. But the only root of this equation is $y = 0$. This means that $y_j \to 0$ as $j \to \infty$ and $\sum_{i=0}^{j} y_i = o(j)$.

Expanding $f(1 + z)$ into the Taylor series, we obtain

$$y_j = y_{j+1} + \frac{B}{2}y_{j+1}^2 + \frac{C}{6}y_{j+1}^3 + O(y_{j+1}^4). \tag{19}$$

From this relation it follows that

$$\frac{y_{j+1}}{y_j} = \frac{1}{1 + By_{j+1}/2 + O(y_{j+1}^2)} = 1 - \frac{By_{j+1}}{2} + O(y_{j+1}^2) = 1 - \frac{By_j}{2} + O(y_j^2). \tag{20}$$

Dividing both parts of (19) by $y_j y_{j+1}$ and making use of (20) we obtain

$$\frac{1}{y_{j+1}} = \frac{1}{y_j} + \frac{B}{2} - \left(\frac{B^2}{4} - \frac{C}{6}\right)y_j + O(y_j^2) = \dots = \frac{1}{y_0} + \frac{Bj}{2} - \left(\frac{B^2}{4} - \frac{C}{6}\right)\sum_{i=0}^{j} y_i + O\left(\sum_{i=0}^{j} y_i^2\right). \tag{21}$$

Therefore,

$$\frac{1}{y_j} = \frac{1}{y_0} + \frac{B}{2}j + o(j).$$

This, in its turn, implies that

$$y_j = \frac{y_0}{1 + Bjy_0/2}(1 + o(1)) \tag{22}$$

and

$$\begin{aligned}\sum_{i=0}^{j-1} y_i^2 &= O\left(\sum_{i=0}^{j-1} \frac{y_0^2}{(1 + Biy_0/2)^2}\right) \\ &= O\left(y_0 \int_0^\infty \frac{y_0\,dx}{(1 + Bxy_0/2)^2}\right) = O(y_0).\end{aligned} \tag{23}$$

Now we study the behaviour of $\sum_{i=0}^{j-1} y_i$. With the use of the inequalities

$$\sum_{i=0}^{j-1} \frac{y_0}{1 + Biy_0/2} \le y_0 + \int_0^j \frac{y_0\,dx}{1 + Bxy_0/2} = y_0 + \frac{2}{B}\ln(1 + Bjy_0/2), \tag{24}$$

$$\sum_{i=0}^{j-1} \frac{y_0}{1 + Biy_0/2} \ge \int_0^j \frac{y_0\,dx}{1 + Bxy_0/2} = \frac{2}{B}\ln(1 + Bjy_0/2), \tag{25}$$

we arrive at the relation

$$\sum_{i=0}^{j-1} y_i = \sum_{i=0}^{j-1} \frac{y_0}{1 + Biy_0/2}(1 + o(1)) = O(\ln(1 + Bjy_0/2)).$$

From (21) and the last equality we obtain

$$\frac{1}{y_j} = \frac{1}{y_0} + \frac{B}{2}j + O(\ln(1 + Bjy_0/2)).$$

Hence it follows that

$$y_j = \frac{y_0}{1 + Bjy_0/2} + O\left(\frac{y_0^2}{(1 + Bjy_0/2)^2}\ln(1 + Bjy_0/2)\right).$$

With the use of this relation we estimate $\sum_{i=0}^{j-1} y_i$ with the accuracy required to prove the lemma:

$$\sum_{i=0}^{j-1} y_i = \sum_{i=0}^{j-1} \frac{y_0}{1 + Biy_0/2} + O\left(y_0^2 \sum_{i=0}^{j-1} \frac{\ln(1 + Biy_0/2)}{(1 + Biy_0/2)^2}\right).$$

It is not difficult to see that the second term in the right-hand side of the preceding equality is $O(y_0)$.

Making use of (24) and (25) again, we obtain

$$\sum_{i=0}^{j-1} y_i = \frac{2}{B}\ln(1 + Bjy_0/2) + O(y_0). \tag{26}$$

Using (21), (23), and (26), we arrive at

$$\frac{1}{y_j} = \frac{1}{y_0} + \frac{B}{2}j - \left(\frac{B}{2} - \frac{C}{3B}\right)\ln(1 + Bjy_0/2) + O(y_0).$$

The lemma is thus proved.

We set $r_0 = 1 + y_n$ and for $i = 1, \dots, n$ define the probability generating function

$$g_j(s) = \frac{f(sf_{j-1}(r_0))}{f_j(r_0)}.$$

We introduce

$$A_j = g_j'(1), \quad A(j) = A_1 \dots A_j, \quad B_j = g_j''(1), \quad T(l) = \sum_{j=0}^{l-1} \frac{B_{j+1}}{2A_{j+1}} A(j).$$

Lemma 2. *If $y_0 \to 0$, then*

$$A(j) = \left(\frac{1 + Bny_0/2}{1 + B(n-j)y_0/2}\right)^2 (1 + O(y_0)), \tag{27}$$

$$T(l) = \frac{Bl(1 + Bny_0/2)}{2(1 + B(n-l)y_0/2)}(1 + O(y_0)) \tag{28}$$

uniformly in $1 \le j \le n$ and $1 \le l \le n$.

Proof. From the definitions of $A(j)$ and $g_j(s)$ it immediately follows that

$$A(j) = \frac{r_0}{f_j(r_0)} \prod_{i=0}^{j-1} f'(1 + y_{n-i}) = \frac{r_0}{f_j(r_0)} \exp\left\{\sum_{i=0}^{j-1} \ln f'(1 + y_{n-i})\right\}.$$

With the use of the relations

$$f'(1+z) = 1 + Bz + O(z^2), \qquad \ln(1+t) = t + O(t^2),$$

we obtain

$$A(j) = \frac{r_0}{f_j(r_0)} \exp\left\{B \sum_{i=0}^{j-1} y_{n-i} + O\left(\sum_{i=0}^{j-1} y_{n-i}^2\right)\right\}.$$

By virtue of (23) and (26),

$$A(j) = \frac{r_0}{f_j(r_0)} \exp\{2\ln(1 + Bny_0) - 2\ln(1 + B(n-j)y_0)) + O(y_0)\}$$

$$= \frac{r_0}{f_j(r_0)} \left(\frac{1 + Bny_0/2}{1 + B(n-j)y_0/2}\right)^2 (1 + O(y_0)).$$

It is not difficult to see that

$$\frac{r_0}{f_j(r_0)} = \frac{1+y_n}{1+y_{n-j}} = 1 + O(y_0).$$

The two last relations imply (27).

Let us turn to the proof of (28). It is obvious that

$$B_i = B(1+O(y_0)), \qquad A_i = 1 + O(y_0)$$

uniformly in i. Therefore,

$$T(l) = \left(\frac{B}{2}\sum_{j=0}^{l-1} A(j)\right)(1+O(y_0)).$$

Using (27), we obtain

$$T(l) = \sum_{j=0}^{l-1} \frac{B}{2}\left(\frac{1+Bny_0/2}{1+B(n-j)y_0/2}\right)^2 (1+O(y_0)). \tag{29}$$

Furthermore, since the function $B/(2(1+Bxy_0/2)^2)$ decreases, the inequalities

$$\int_{n-l+1}^{n+1} \frac{B\,dx}{2(1+Bxy_0/2)^2} \le \sum_{j=0}^{l-1}(1+B(n-j)y_0/2)^{-2} \le \int_{n-l}^{n} \frac{B\,dx}{2(1+Bxy_0/2)^2}$$

are true. It is not difficult to see that each integral in the preceding inequality is

$$\frac{Bl}{2(1+Bny_0/2)(1+B(n-l)y_0/2)}(1+O(y_0)).$$

Therefore,

$$\sum_{j=0}^{l-1}(1+B(n-j)y_0/2)^{-2} = \frac{Bl}{2(1+Bny_0/2)(1+B(n-l)y_0/2)}(1+O(y_0)). \tag{30}$$

From (29) and (30) we arrive at the required relation.

Let

$$\rho(s) = \sum_{k=0}^{\infty} \rho_k s^k;$$

we set

$$\|\rho\|_1 = \sum_{k=0}^{\infty} |\rho_k|.$$

It is obvious that $\|\cdot\|_1$ possesses all properties of a norm.

Lemma 3. *Let $\rho(s)$ be a probability generating function. We set $a = \rho'(1)$, $b = \rho''(1)$. Then*

$$\frac{1}{1-\rho(s)} = \frac{1}{a(1-s)} + \frac{b}{2a^2} + (\delta(s) - \eta(s))\frac{(1-s)^2}{1-\rho(s)}, \tag{31}$$

where

$$\delta(s) = \frac{b(\rho(s) - 1 - a(s-1))}{2a^2(s-1)^2}, \qquad \eta(s) = \frac{\rho(s) - 1 - a(s-1) - b(s-1)^2/2}{a(s-1)^3}.$$

If we introduce the extra constraint $c = \rho'''(1) < \infty$, then

$$\|\delta(s) - \eta(s)\|_1 \le \frac{b^2}{4a^2} + \frac{c}{6a}.$$

Proof. From the equality

$$\frac{1}{1-\rho(s)} - \frac{1}{a(1-s)} = \left(\frac{\rho(s) - 1 - a(s-1)}{a(s-1)^2}\right)\frac{(1-s)}{1-\rho(s)} \tag{32}$$

it follows that

$$\frac{1}{1-\rho(s)} - \frac{1}{a(1-s)} = \frac{b(1-s)}{2a(1-\rho(s))} - \eta(s)\frac{(1-s)^2}{1-\rho(s)}. \tag{33}$$

Furthermore, (32) yields

$$\frac{1-s}{1-\rho(s)} = \frac{1}{a} + \left(\frac{\rho(s) - 1 - a(s-1)}{a(s-1)^2}\right)\frac{(1-s)^2}{1-\rho(s)}. \tag{34}$$

Substituting this into (33), we finally arrive at relation (31).

It is easy to verify that

$$\frac{\rho(s) - 1 - a(s-1)}{(s-1)^2} = \sum_{k=2}^{\infty} \rho_k \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} s^j.$$

Therefore,

$$\left\| \frac{\rho(s) - 1 - a(s-1)}{(s-1)^2} \right\|_1 = \frac{b}{2}. \tag{35}$$

Similarly we find that

$$\left\| \frac{\rho(s) - 1 - a(s-1) - b(s-1)^2}{(s-1)^3} \right\|_1 = \frac{c}{6}. \tag{36}$$

Inequalities (35) and (36) yield the estimate

$$\|\delta(s) - \eta(s)\|_1 \le \|\delta(s)\|_1 + \|\eta(s)\|_1 = \frac{b^2}{4a^2} + \frac{c}{6a},$$

which proves the lemma.

We set

$$G_i(s) = g_i \circ \ldots \circ g_1(s) = f_i(r_0 s)/f_i(r_0), \qquad q_i(s) = 1 - G_i(s).$$

In what follows we assume that $y_0 \to 0$.

Lemma 4. *For any $i \geq 1$, the inequality*

$$\|q_i(s)\|_1 \leq 2(Q_i + y_{n-i}) \tag{37}$$

is true. From this bound it follows that $q_i(s) \to 0$ as $i \to \infty$ uniformly in s inside the unit disk.

Proof. Let

$$\rho(s) = \sum_{k=0}^{\infty} \rho_k s^k$$

be a probability generating function. It is easy to check that

$$\|1 - \rho(s)\|_1 = (1 - \rho_0) + \sum_{k=1}^{\infty} \rho_k = 2(1 - \rho(0)).$$

Applying this equality to $G_i(s)$, we obtain

$$\|q_i(s)\|_1 = 2\left(1 - \frac{f_i(0)}{f_i(r_0)}\right) = 2\frac{f_i(r_0) - f_i(0)}{f_i(r_0)} \leq 2(Q_i + f_i(r_0) - 1).$$

Since $f_(r_0) = 1 + y_{n-i}$, we arrive at inequality (37).

The convergence of $\|q_i(s)\|_1$ to zero follows from the facts that $Q_i \to 0$ as $i \to \infty$ and $y_0 \to 0$. The uniform inside the unit disk convergence of $q_i(s)$ to zero follows from the inequality $\sup_{|s| \leq 1} |q_i(s)| \leq \|q_i(s)\|_1$.

Lemma 5. *Let* $\gcd\{k\colon p_k > 0\} = 1$. *Then there exists N such that*

$$q_j(s) = \frac{A(j)}{(1-s)^{-1} + T(j)} \left(1 - \frac{R_N(s) + \sum_{i=N}^{j-1} A(i) q_i(s) C_i(s)}{(1-s)^{-1} + T(j)}\right)^{-1} \tag{38}$$

for $j \geq N$, $|s| \leq 1$, where $C_i(s)$, $R_N(s)$ are some analytic in the unit disk functions, and $\|C_i(s)\|_1 \leq C$, $\|R_N(s)\|_1 < \infty$.

Proof. Substituting $\rho = g_{j+1}$, $s = G_j$ into (31), we obtain the equality

$$\frac{1}{q_{j+1}(s)} = \frac{1}{A_{j+1} q_j(s)} + \frac{B_{j+1}}{2A_{j+1}^2} - d_{j+1}(s) \frac{q_j^2(s)}{A_{j+1} q_{j+1}(s)},$$

where

$$d_{j+1}(s) = -A_{j+1}(\delta_{j+1}(1 - q_j(s)) - \eta_{j+1}(1 - q_j(s))).$$

Multiplying both sides of this equality by $A(j+1)$, we arrive at the recurrence relation

$$b_{j+1}(s) = b_j(s) + \frac{B_{j+1}}{2A_{j+1}}A(j) - d_{j+1}(s)A(j)\frac{q_j^2(s)}{q_{j+1}(s)},$$

where $b_j(s) = A(j)/q_j(s)$. From this equality we easily derive the following expansion of the function $b_j(s)$:

$$\begin{aligned} b_j(s) &= b_N(s) + T(j) - T(N) - \sum_{i=N}^{j-1} A(i)q_i(s)C_i(s) \\ &= \frac{1}{1-s} + T(j) - R_N(s) - \sum_{i=N}^{j-1} A(i)q_i(s)C_i(s), \end{aligned}$$

where

$$C_i(s) = d_{i+1}(s)q_i/q_{i+1}(s), \qquad R_N(s) = (1-s)^{-1} + T(N) - b_N(s),$$

N is an arbitrary positive integer less than j. This proves representation (38).

It remains to prove that the functions $C_i(s)$, $R_N(s)$ are analytic and their norms are bounded.

From Lemma 3 it follows that

$$\|d_{j+1}\|_1 \le A_{j+1}(\|\delta_{j+1}\|_1 + \|\eta_{j+1}\|_1) \le \left(\frac{(g''_{j+1}(1))^2}{4A_{j+1}} + \frac{g'''_{j+1}(1)}{6}\right). \tag{39}$$

From this relation it follows that $\|d_{j+1}\|_1$ are uniformly bounded.

If we assume that $\gcd\{k\colon \rho_k > 0\} = 1$, then the function $(1-\rho(s))/(1-s)$ has no zeros inside the unit disk. If s_0 were a zero of this function, then $\rho(s_0) = 1$. Therefore, $|s_0| = 1$ and $s_0^k = 1$ for any k such that $\rho_k > 0$. From the aperiodicity condition it follows that the only point which satisfies these conditions is $s_0 = 1$. But this point does not make the function $(1-\rho(s))/(1-s)$ vanish because

$$\left.\frac{1-\rho(s)}{1-s}\right|_{s=1} = \sum_{k=1}^{\infty} \rho_k(1+\ldots+s^{k-1})|_{s=1} = a.$$

By virtue of the Tauberian theorem due to Wiener (see, e.g. [8]), the function $(1-s)/(1-\rho(s))$ is analytic and $\|(1-s)/(1-\rho(s))\|_1 < \infty$.

From aperiodicity of the sequence p_k it follows that the coefficients of the function $g_j(s)$ also form an aperiodic sequence. Therefore,

$$\left\|\frac{q_j}{q_{j+1}}\right\|_1 \le \left\|\frac{1-s}{1-g_{j+1}(s)}\right\|_1 < \infty.$$

Furthermore, relation (34) yields the estimate

$$\left\|\frac{q_j}{q_{j+1}}\right\|_1 \le \frac{1}{A_{j+1}} + \frac{B_{j+1}}{2A_{j+1}}\left\|\frac{q_j}{q_{j+1}}\right\|_1 \|q_j\|_1.$$

Making use of Lemma 4, we conclude that

$$\left\| \frac{q_j}{q_{j+1}} \right\|_1 \le 2 \tag{40}$$

for all j exceeding a certain threshold value and for sufficiently small y_0.

From (39) and (40) it follows that the norms of $C_i(s)$ are bounded for all i no less than a certain N.

Let us consider the function $R_N(s)$. It is clear that

$$\|R_N(s)\|_1 \le T(N) + A(N) \left\| \frac{1}{q_N(s)} - \frac{1}{A(N)(1-s)} \right\|_1 .$$

Setting $\rho(s) = G_N(s)$ in (32), we obtain

$$\frac{1}{q_N(s)} - \frac{1}{A(N)(1-s)} = \left(\frac{G_N(s) - 1 - A(N)(s-1)}{A(N)(s-1)^2} \right) \frac{1-s}{q_N(s)}.$$

Making use of equality (35), we find that

$$\left\| \frac{G_N(s) - 1 - A(N)(s-1)}{A(N)(s-1)^2} \right\|_1 = \frac{G''_N(1)}{2A(N)} = \frac{T(N)}{2A(N)}.$$

Therefore,

$$\left\| \frac{1}{q_N(s)} - \frac{1}{A(N)(1-s)} \right\|_1 \le \frac{T(N)}{2} \left\| \frac{1-s}{q_N(s)} \right\|_1 .$$

The boundedness of $\|(1-s)/q_N(s)\|_1$ follows from aperiodicity of the coefficients of the function $G_N(s)$ and the Wiener theorem mentioned above. Finally, we see that $R_N(s)$ is bounded in the norm.

Lemma 6. *As $i \to \infty$,*

$$q_i(s) \left(\frac{(1-s)^{-1} + T(i)}{A(i)} \right) \to 1 \tag{41}$$

uniformly in all s inside the unit disk.

Proof. It is clear that in order to prove the lemma it suffices to check that the function

$$\frac{R_N(s) + \sum_{i=N}^{j-1} A(i) q_i(s) C_i(s)}{(1-s)^{-1} + T(j)}$$

converges to zero uniformly in the unit disk. Let α_i, β_i be numerical sequences satisfying the conditions

$$\sum_{i=1}^{n} \alpha_i \to \infty, \qquad \beta_i \to 0.$$

Then, as we easily see,

$$\frac{\sum_{i=1}^{n} \alpha_i \beta_i}{\sum_{i=1}^{n} \alpha_i} \to 0.$$

From the definition of $T(j)$ it follows that

$$\sum_{i=N}^{j-1} A(i) = O(T(j)).$$

By virtue of Lemma 4, $q_i(0) \to 0$. From the abovesaid it follows that

$$\frac{\sum_{i=N}^{j-1} A(i)q_i(0)}{T(j)} \to 0. \tag{42}$$

It is obvious that

$$\left\| \frac{R_N(s) + \sum_{i=N}^{j-1} A(i)q_i(s)C_i(s)}{(1-s)^{-1} + T(j)} \right\|_1 \le \frac{\|R_N(s)\|_1 + \sum_{i=N}^{j-1} A(i)\|q_i(s)\|_1 \|C_i(s)\|_1}{T(j)}.$$

Since $\|R_N(s)\|_1 < \infty$ and $T(j) \to \infty$, the relation $\|R_N(s)\|_1/T(j) \to 0$ is true. Making use of Lemmas 4 and 5, we obtain

$$\frac{1}{T(j)} \sum_{i=N}^{j-1} A(i)\|q_i(s)\|_1 \|C_i(s)\|_1 \le \frac{2C}{T(j)} \sum_{i=N}^{j-1} A(i)q_i(0).$$

In view of (42), the right-hand side of the last inequality tends to zero. Thus,

$$\left\| \frac{R_N(s) + \sum_{i=N}^{j-1} A(i)q_i(s)C_i(s)}{(1-s)^{-1} + T(j)} \right\|_1 \to 0.$$

This, in turn, means that the function

$$\frac{R_N(s) + \sum_{i=N}^{j-1} A(i)q_i(s)C_i(s)}{(1-s)^{-1} + T(j)}$$

converges to zero uniformly in s inside the unit disk. The lemma is thus proved.

Let $\rho(s)$ be a power series. Let $a_k[\rho(s)]$ denote the kth coefficient of this series.

Lemma 7. *Let $\rho(s)$ be a probability generating function and $\rho'(1) < \infty$. Then the inequality*

$$a_l[\rho(s)] \le \frac{1}{l} \int_{-a}^{a} |\rho'(e^{it})|\, dt \tag{43}$$

holds for any $a > 0$.

Proof. It is obvious that the function $\rho'(s)/\rho'(1)$ is a probability generating function. The following bound for the concentration function is well known [4]:

$$\sup_x \mathbf{P}(X = x) \le (96/95)^2 (la)^{-1} \int_{-a}^{a} |\varphi(t)|\, dt.$$

where $\varphi(t)$ is the characteristic function of the variable X, $a > 0$.

Applying this bound to the random variable whose distribution corresponds to the generating function $\rho'(s)/\rho'(1)$, we obtain

$$\sup_l a_l[\rho'(s)/\rho'(1)] \le (96/95)^2 a^{-1} \int_{-a}^{a} |\rho'(e^{it})/\rho'(1)|\, dt.$$

It is not difficult to see that

$$a_l[\rho(s)] = \frac{1}{l}\, a_l[\rho'(s)]$$

for any $l \ge 1$; the two last relations prove the lemma.

Lemma 8. *There exists a constant* M_1 *such that*

$$|G_k'(s)| \le M_1 A(k) \exp\left\{-\sum_{i=1}^{k} \frac{B_i}{A_i} \Re(q_{i-1}(s))\right\}. \tag{44}$$

Proof. From the definition of $G_k(s)$ it follows that

$$G_k'(s) = \prod_{i=1}^{k} g_i'(G_{i-1}(s)) = \exp\left\{\sum_{i=1}^{k} \ln g_i'(G_{i-1}(s))\right\}.$$

Now, by virtue of the equalities

$$\begin{aligned} g_i'(s) &= g_i'(1) + B_i(s-1) + O((s-1)^2), \\ \ln g_i'(s) &= \ln g_i'(1) + \ln(1 + B_i(s-1)/g_i'(1) + O((s-1)^2)) \\ &= \ln g_i'(1) + \frac{B_i}{A_i}(s-1) + O((s-1)^2) \end{aligned}$$

we obtain

$$G_k'(s) = A(k) \exp\left\{-\sum_{i=1}^{k} \frac{B_i}{A_i} q_{i-1}(s) + O\left(\sum_{i=1} q_{i-1}^2(s)\right)\right\}. \tag{45}$$

Making use of inequality (37), we see that

$$\sum_{i=1}^{k} |q_{i-1}(s)|^2 \le 4 \sum_{i=0}^{k-1} Q_i^2 + 4 \sum_{j=0}^{k-1} y_j^2. \tag{46}$$

Boundedness of the former sum in the right-hand side of (46) follows from the fact that $Q_i = O(1/i)$, while boundedness of the latter sum, from (23). Hence there exists a constant M_1 such that

$$|G_k'(s)| \le M_1 A(k) \left| \exp\left\{-\sum_{i=1}^{k} \frac{B_i}{A_i} q_{i-1}(s)\right\}\right|.$$

Taking into account the formula $|e^z| = e^{\Re z}$, we arrive at the inequality we set up to prove.

Lemma 9. *There exists a positive integer N such that the inequality*

$$\sum_{j=1}^{k} \frac{B_j}{A_j} \Re(q_{j-1}(e^{it})) \geq \frac{3}{4} \ln\left(1 + \frac{(2T(k)+1)^2(1-\cos t)^2}{\sin^2 t}\right) - a(N) \tag{47}$$

is true for all $k > N$, where $a(N)$ is some constant.

Proof. By virtue of Lemma 6, as $j \to \infty$,

$$q_j(s)\left(\frac{(1-s)^{-1}+T(j)}{A(j)}\right) \to 1$$

uniformly in s inside the unit disk. This means that there exists N such that for all $j \geq N$

$$\Re(q_j(s)) \geq \frac{3}{4}\Re\left[\frac{A(j)}{(1-s)^{-1}+T(j)}\right] - \frac{1}{4}\left|\Im\left[\frac{A(j)}{(1-s)^{-1}+T(j)}\right]\right|.$$

We set $s = e^{it}$. Making use of the relation

$$\frac{1}{1-s} = \frac{1}{(1-\cos t) - i\sin t} = \frac{1}{2} + \frac{i\sin t}{2(1-\cos t)},$$

we obtain

$$\Re\left[\frac{A(j)}{(1-s)^{-1}+T(j)}\right] = \frac{2A(j)(2T(j)+1)}{(2T(j)+1)^2 + \sin^2 t(1-\cos t)^{-2}},$$

$$\Im\left[\frac{A(j)}{(1-s)^{-1}+T(j)}\right] = -\frac{2A(j)(1-\cos t)\sin^{-1} t}{1+(2T(j)+1)^2(1-\cos t)^2\sin^{-2} t}.$$

Therefore,

$$\sum_{j=N}^{k-1} \frac{B_{j+1}}{A_{j+1}}\Re(q_j(s)) \geq \frac{3}{4}\sum_{j=N}^{k-1}\frac{B_{j+1}}{A_{j+1}}\frac{2A(j)(2T(j)+1)}{(2T(j)+1)^2+\sin^2 t(1-\cos t)^{-2}}$$
$$- \frac{1}{4}\sum_{j=0}^{k-1}\frac{B_{j+1}}{A_{j+1}}\frac{2A(j)(1-\cos t)|\sin^{-1} t|}{1+(2T(j)+1)^2(1-\cos t)^2\sin^{-2} t}. \tag{48}$$

By the definition of $T(j)$,

$$\frac{B_{j+1}}{2A_{j+1}}A(j) = T(j+1) - T(j) = \Delta T(j).$$

Hence,

$$\sum_{j=N}^{k-1}\frac{4(2T(j)+1)\Delta T(j)}{(2T(j)+1)^2+\sin^2 t(1-\cos t)^{-2}} \geq \int_{2T(N)+1}^{2T(k)+1}\frac{2x\,dx}{x^2+\sin^2 t(1-\cos t)^{-2}}$$
$$= \ln\left(1+\frac{(2T(k)+1)^2(1-\cos t)^2}{\sin^2 t}\right) - \ln\left(1+\frac{(2T(N)+1)^2(1-\cos t)^2}{\sin^2 t}\right). \tag{49}$$

Similarly we find that

$$\frac{1}{4}\sum_{j=0}^{k-1}\frac{B_{j+1}}{A_{j+1}}\frac{2A(j)(1-\cos t)|\sin^{-1}t|}{1+(2T(j)+1)^2(1-\cos t)^2\sin^{-2}t} \le \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}. \tag{50}$$

Now the validity of the lemma immediately follows from (48), (49), and (50).

Lemma 10. *There exists a constant M_2 such that,*

$$|a_l[q_k(s)]| \le M_2\frac{A(k)}{lT(k)} \tag{51}$$

for all $k, l \ge 1$.

Proof. It is obvious that $a_l[q_k(s)] = -a_l[G_k(s)]$. Therefore, it suffices to prove that (51) is true for the coefficients of the function $G_k(s)$.

From Lemmas 8 and 9 it follows that

$$|G_k'(e^{it})| \le c_1 A(k)T^{-3/2}(k)\frac{|\sin t|^{3/2}}{(1-\cos t)^{3/2}}, \tag{52}$$

where c_1 is some constant.

Setting $\rho = G_k$, $a = \pi/2$ in inequality (43), we obtain

$$a_l[G_k(s)] \le \frac{1}{l}\left(\int_{1/T(k)<|t|<\pi/2}|G_k'(e^{it})|\,dt + \int_{|t|<1/T(k)}|G_k'(e^{it})|\,dt\right). \tag{53}$$

The obvious bound $|G_k'(e^{it})| \le G_k'(1) = A(k)$ implies the inequality

$$\int_{|t|<1/T(k)}|G_k'(e^{it})|\,dt \le 2\frac{A(k)}{T(k)}. \tag{54}$$

It is not difficult to see that

$$\frac{|\sin t|}{|1-\cos t|} \le \frac{2}{|t|}.$$

Applying this bound to the right-hand side of (52), we conclude that

$$|G_k'(e^{it})| \le 4c_1A(k)T^{-3/2}(k)|t|^{-3/2}.$$

Therefore,

$$\int_{1/T(k)<|t|<\pi/2}|G_k'(e^{it})|\,dt \le 4c_1A(k)T^{-3/2}(k)\int_{1/T(k)<|t|<\pi/2}|t|^{-3/2}dt \le c_2\frac{A(k)}{T(k)}. \tag{55}$$

Combining (53), (54), and (55), we arrive at the inequality we wished to prove.

Let $\{\tau_k\}_1^\infty$ be a sequence of positive numbers satisfying the condition

$$K_\tau = \sup_{n\ge 1}\sup_{k\ge n/2}\left(\frac{\tau_k}{\tau_n}\right) < \infty.$$

For any $\rho(s)$ such that $\|\rho(s)\|_1 < \infty$ we set

$$P_\tau(\rho) = \sup_{n\ge 1} |\rho_n|/\tau_n.$$

In [9], the inequality

$$P_\tau(\rho_1\rho_2) \le K_\tau(P_\tau(\rho_1)\|\rho_2\|_1 + P_\tau(\rho_2)\|\rho_1\|_1) \tag{56}$$

was established. With the use of this bound, we arrive at the following assertion.

Lemma 11. *Let*

$$\lambda(t) = \sum_{i=0}^{\infty} \lambda_i t^i$$

be a series with non-negative coefficients which converges for $|t| < t_0$. *Then*

$$P_\tau(\lambda(\rho)) \le \lambda'(K_\tau\|\rho\|_1)P_\tau(\rho) \tag{57}$$

for any $\rho(s)$ *such that* $K_\tau\|\rho\|_1 < t_0$.

Proof. With the use of induction, from inequality (56) we derive the bound

$$P_\tau(\rho^i) \le H_i\|\rho\|_1^{i-1}P_\tau(\rho),$$

where H_i are defined as follows:

$$H_1 = 1, \qquad H_{i+1} = K_\tau(1 + H_i).$$

It is clear that $K_\tau \ge 1$. Hence,

$$H_{i+1} \le K_\tau^i + K_\tau H_i \le \ldots \le (i+1)K_\tau^i.$$

Therefore,

$$P_\tau(\rho^i) \le iK_\tau^{i-1}\|\rho\|_1^{i-1}P_\tau(\rho).$$

On the other hand,

$$P_\tau(\lambda(\rho)) \le \sum_{i=1}^{\infty}\lambda_i P_\tau(\rho^i).$$

Combining the two last inequalities proves the lemma.

In what follows we set $\tau_n = n^{-1}$. It is clear that $K_\tau = 2$.

Lemma 12. *There exists a constant M_3 such that the inequality*

$$P_\tau(C_i) \le M_3 \frac{A(i)}{T(i)} \tag{58}$$

is true for all i no less than a certain N.

Proof. Let $\delta(s)$ be defined as in Lemma 3. We assume that the generating function $\rho(s)$ that determines $\delta(s)$ converges in the disk of radius $R = 1 + t_0$. It is easy to verify that

$$\delta(s) = \frac{b}{2a^2} \sum_{i=2}^{\infty} \frac{\rho^{(i)}(1)}{i!}(s-1)^{i-2}.$$

Let $q(s)$ be a power series of finite norm which satisfies the condition $\|q\|_1 < t_0/K_\tau$. Then, by virtue of Lemma 11,

$$\begin{aligned} P_\tau(\delta(1-q)) &\le P_\tau(q)\frac{b}{2a^2}\sum_{i=3}^{\infty}\frac{\rho^{(i)}(1)}{i!}(i-2)(K_\tau\|q\|_1)^{i-3} \\ &\le P_\tau(q)\frac{b}{2a^2}\rho^{(3)}(1+K_\tau\|q\|_1). \end{aligned} \tag{59}$$

From Lemma 10 it follows that

$$P_\tau(q_i) \le M_2 \frac{A(i)}{T(i)}. \tag{60}$$

Since $\|q_i\|_1$ and y_0 tend to zero, the inequality

$$(1 + 2\|q_i\|_1)(1 + y_0) < \frac{R+1}{2}$$

holds for all $i \ge N$. It is easy to see that

$$g_{i+1}^{(3)}(1+x) \le (1+y_0)^2 f^{(3)}((1+x)(1+y_0)).$$

Setting $\rho = g_{i+1}$, $q = q_i$ in (59) and taking into account the two last bounds, we obtain

$$P_\tau(\delta_{i+1}(1-q_i)) \le P_\tau(q_i)\frac{B_{i+1}}{2A_{i+1}^2}(1+y_0)^2 f^{(3)}((R+1)/2).$$

Since

$$B_{i+1} = B(1 + O(y_0)), \qquad A_{i+1} = 1 + O(y_0)$$

and inequality (60) is true, there exists a constant c_1 such that

$$P_\tau(\delta_{i+1}(1-q_i)) \le c_1 \frac{A(i)}{T(i)}. \tag{61}$$

The same reasoning yields

$$P_\tau(\eta_{i+1}(1-q_i)) \le c_2 \frac{A(i)}{T(i)}.$$

From the two last relations it follows that

$$P_\tau(d_{i+1}) \le A_{i+1} P_\tau(\delta_{i+1}(1-q_i)) + A_{i+1} P_\tau(\eta_{i+1}(1-q_i)) \le c_3 \frac{A(i)}{T(i)}. \tag{62}$$

Further, setting $\rho = g_{i+1}$, $s = q_i$ in (34), we obtain

$$\frac{q_i(s)}{q_{i+1}(s)} = \frac{1}{A_{i+1}} + \frac{2A_{i+1}}{B_{i+1}} \delta_{i+1}(1-q_i(s)) \frac{q_i^2(s)}{q_{i+1}(s)}.$$

Therefore,

$$\frac{q_i(s)}{q_{i+1}(s)} = \frac{1}{A_{i+1}} \sum_{l=0}^{\infty} \frac{2^l A_{i+1}^l}{B_{i+1}^l} \delta_{i+1}^l (1-q_i(s)) q_i^l(s).$$

Applying inequality (56) to the functions δ_{i+1} and $q_i(s)$ and taking (61) into account, we arrive at the bound

$$P_\tau(\delta_{i+1} q_i) \le c_4 \frac{A(i)}{T(i)}.$$

From this bound and Lemma 11 we find that

$$P_\tau(q_i/q_{i+1}) \le c_5 \frac{A(i)}{T(i)} \left(1 - \frac{2A_{i+1}}{B_{i+1}} \|\delta_{i+1}\|_1 \|q_i\|_1\right)^{-2} \le c_6 \frac{A(i)}{T(i)}. \tag{63}$$

Applying inequality (56) to the functions d_{i+1} and q_i/q_{i+1}, with the use of relations (62) and (63) we complete the proof of the lemma.

3. PROOF OF THEOREM 1

Without loss of generality we assume that $k > Bn/2$. Let

$$y_0 = \frac{4k}{B^2 n^2} - \frac{2}{Bn}.$$

From the hypotheses of the theorem it follows that $y_0 \to 0$ as $n \to \infty$. It is not difficult to verify that

$$\begin{aligned} \mathbf{P}(Z_n = k) &= a_k[f_n(s)] = r_0^{-k} f_n(r_0) a_k[f_n(r_0 s)/f_n(r_0)] \\ &= r_0^{-k}(1+y_0) a_k[G_n(s)] = -r_0^{-k}(1+y_0) a_k[q_n(s)]. \end{aligned} \tag{64}$$

Setting $y_0 = 4k/B^2n^2 - 2/Bn$ in Lemma 1, we obtain

$$y_n = \frac{2}{Bn} - \frac{1}{k} + \frac{B}{2}\gamma \left(\frac{2}{Bn} - \frac{1}{k}\right)^2 \ln(2k/Bn) + O(n^{-2}).$$

Removing the brackets, with the use of the inequality $\ln x < x$, we arrive at the relation

$$y_n = \frac{2}{Bn} - \frac{1}{k} + \frac{2}{B}\gamma \frac{1}{n^2} \ln(k/n) + O(n^{-2}).$$

Therefore,

$$r_0^{-k} = (1+y_n)^{-k} = \exp\left\{-\frac{2k}{Bn} + 1 - \frac{2}{B}\gamma\frac{k}{n^2}\ln(k/n)\right\}(1+O(k/n^2)). \qquad (65)$$

Let us turn to the study of the asymptotic behaviour of $a_l[q_n(s)]$.
Lemma 5 yields

$$q_n(s) = \frac{A(n)}{(1-s)^{-1}+T(n)} + \frac{q_n(s)}{(1-s)^{-1}+T(n)}\left(R_N(s) + \sum_{i=N}^{n-1} A(i)C_i(s)q_i(s)\right).$$

Hence,

$$a_l[q_n(s)] = a_l\left[\frac{A(n)}{(1-s)^{-1}+T(n)}\right] + a_l\left[\frac{q_n(s)}{(1-s)^{-1}+T(n)}\left(R_N(s) + \sum_{i=N}^{n-1} A(i)C_i(s)q_i(s)\right)\right]. \qquad (66)$$

It is easy to check that

$$\frac{1}{(1-s)^{-1}+T(n)} = \frac{1}{1+T(n)} - \frac{1}{(1+T(n))^2}\sum_{j=1}^{\infty}\left(\frac{T(n)}{1+T(n)}\right)^{j-1} s^j.$$

Therefore,

$$\left|a_l\left[\frac{q_n(s)}{(1-s)^{-1}+T(n)}\left(R_N(s) + \sum_{i=N}^{n-1} A(i)C_i(s)q_i(s)\right)\right]\right| \le F_1 + F_2, \qquad (67)$$

where

$$F_1 = \frac{1}{1+T(n)}\left|a_l\left[q_n(s)\left(R_N(s) + \sum_{i=N}^{n-1} A(i)C_i(s)q_i(s)\right)\right]\right|,$$

$$F_2 = \frac{\|q_n(s)\|_1}{(1+T(n))^2}\left(\|R_N(s)\|_1 + \sum_{i=N}^{n-1} A(i)\|C_i(s)\|_1\|q_i(s)\|_1\right).$$

We begin with estimating F_1. Lemmas 10, 12, and inequality (56) imply the bound

$$P_\tau(C_i q_i) \le c_1\frac{A(i)}{T(i)}.$$

Again making use of Lemma 10 and (56), we obtain

$$P_\tau(q_n C_i q_i) \le c_2\left(\|q_i\|_1\frac{A(n)}{T(n)} + \|q_n\|_1\frac{A(i)}{T(i)}\right) = 2c_2\left(q_i(0)\frac{A(n)}{T(n)} + q_n(0)\frac{A(i)}{T(i)}\right).$$

Lemma 6 guarantees the existence of constants c_3, c_4 such that

$$c_3 \frac{A(i)}{T(i)} \le q_i(0) \le c_4 \frac{A(i)}{T(i)} \tag{68}$$

for all i. From the two last relations we find that

$$P_\tau\left(q_n \sum_{i=N}^{n-1} A(i) C_i q_i\right) \le \sum_{i=N}^{n-1} A(i) P_\tau(q_n C_i q_i) \le c_5 \frac{A(n)}{T(n)} \sum_{i=N}^{n-1} A(i) q_i(0).$$

Lemma 3 and relation (7) imply the inequality

$$q_i(0) \le c_6(y_{n-i} + i^{-1}) \le c_6(y_0 + i^{-1}).$$

Since $A(i)$ increases,

$$\begin{aligned} \sum_{i=N}^{n-1} A(i) q_i(0) &\le c_6 y_0 T(n) + c_6 \sum_{i=1}^{n-1} A(i) i^{-1} \\ &\le c_6 y_0 T(n) + c_6 A([n/2]) \ln n + 2c_6 T(n) n^{-1} \le c_7(y_0 T(n) + \ln n), \end{aligned} \tag{69}$$

because $A([n/2])$ is bounded and $n^{-1} \le c_8 y_0$. We thus obtain

$$P_\tau\left(q_n \sum_{i=N}^{n-1} A(i) C_i q_i\right) = O(A(n)(y_0 + T^{-1}(n) \ln n)).$$

Therefore, if $l \ge \alpha T(n)$, then

$$\frac{1}{1+T(n)} \left| a_l \left[q_n(s) \sum_{i=N}^{n-1} A(i) C_i(s) q_i(s) \right] \right| = O\left(\frac{A(n)}{T^2(n)} (y_0 + T^{-1}(n) \ln n) \right) \tag{70}$$

for any $\alpha > 0$.

By virtue of Lemmas 5 and 10,

$$\begin{aligned} a_l[q_n(s) R_N(s)] &\le \|R_N(s)\|_1 \sup_{t \ge l/2} a_t[q_n(s)] + \|q_n(s)\|_1 \sup_{t \ge l/2} a_t[R_N(s)] \\ &\le R \frac{2A(n)}{lT(n)} + c_{15} \frac{A(n)}{T(n)} \sup_{t \ge l/2} a_t[R_N(s)]. \end{aligned}$$

By the definition of $R_N(s)$,

$$a_l[R_N(s)] = A(N) a_l \left[\frac{1}{1 - G_N(s)} - \frac{1}{A(N)(1-s)} \right].$$

The fact that $G''_N(1) < \infty$ and the result due to Gelfond [10, 11] yield

$$\sup_{t \ge l/2} a_t[R_N(s)] = O(T^{-1}(n))$$

for $l \geq \alpha T(n)$. Therefore,

$$\frac{1}{1+T(n)} a_l[q_n(s) R_N(s)] = O\left(\frac{A(n)}{T^3(n)}\right) \tag{71}$$

for $l \geq \alpha T(n)$.

From (70) and (71) it follows that

$$F_1 = O\left(\frac{A(n)}{T^2(n)}(y_0 + T^{-1}(n)\ln n)\right), \tag{72}$$

if $l \geq \alpha T(n)$ for some $\alpha > 0$.

Let us estimate F_2. By virtue of Lemmas 3, 5, and relation (69),

$$\sum_{k=N}^{n-1} A(i)\|C_i(s)\|_1 \|q_i(s)\|_1 \leq 2C \sum_{k=N}^{n-1} A(i) q_i(0) = O(T(n)(y_0 + T^{-1}(n)\ln n)).$$

In view of the equality $\|q_n(s)\|_1 = 2q_n(0)$ and (68),

$$\|q_n(s)\|_1 \left(\|R_N(s)\|_1 + \sum_{k=N}^{n-1} A(i)\|C_i(s)\|_1 \|q_i(s)\|_1 \right) = O(A(n)(y_0 + \ln n T^{-1}(n))),$$

which yields

$$F_2 = O\left(\frac{A(n)}{T^2(n)}(y_0 + T^{-1}(n)\ln n)\right). \tag{73}$$

From (66), (72), (67), and (73) it follows that the relation

$$a_l[q_n(s)] = -\frac{A(n)}{T^2(n)}\left(\frac{T(n)}{1+T(n)}\right)^{l-1} + O\left(\frac{A(n)}{T^2(n)}(y_0 + T^{-1}(n)\ln n)\right)$$

is true for all $l \geq \alpha T(n)$. Hence,

$$a_l[q_n(s)] = -\frac{A(n)}{T^2(n)} \exp\left\{-\frac{l}{T(n)}\right\}(1 + O(y_0 + T^{-1}(n)\ln n)) \tag{74}$$

holds uniformly in $\alpha T(n) \leq l \leq \beta T(n)$, where α, β, $\alpha < \beta$, are arbitrary real numbers.

From the definition of y_0 and formulas (27), (28) it follows that

$$T(n) = k(1 + O(y_0)), \qquad \frac{A(n)}{T^2(n)} = \frac{4}{B^2 n^2}(1 + O(y_0)). \tag{75}$$

Setting $l = k$ in (74), we obtain

$$a_k[q_n(s)] = -\frac{4}{B^2 n^2} e^{-1}(1 + O(kn^{-2} + k^{-1}\ln n)).$$

This equality and formulas (64), (65) prove the theorem.

4. PROOF OF THEOREM 2

First we assume that $k \geq \lambda n \ln n$, where λ is chosen so that the relation

$$r_0^{-\lambda n \ln n} = O(n^{-1}) \tag{76}$$

holds for any $k \geq \lambda n \ln n$. The existence of such λ follows from (65).

Furthermore, by virtue of Theorem 1

$$\mathbf{P}(Z_n \geq k) = \sum_{i=k}^{2k-1} \frac{4}{B^2 n^2} \left(\exp\left\{ -\frac{2i}{Bn} - \frac{2\gamma i}{Bn^2} \ln\left(\frac{i}{n}\right) \right\} \right) (1 + O(kn^{-2})) + \mathbf{P}(Z_n \geq 2k). \tag{77}$$

Let $\gamma > 0$. Then

$$\begin{aligned} \sum_{i=k}^{2k-1} \exp\left\{ -\frac{2i}{Bn} - \frac{2}{B}\gamma \frac{i}{n^2} \ln\left(\frac{i}{n}\right) \right\} &\leq \exp\left\{ -\frac{2k}{Bn} - \frac{2}{B}\gamma \frac{k}{n^2} \ln\left(\frac{k}{n}\right) \right\} \sum_{i=0}^{\infty} e^{-2i/Bn} \\ &= \frac{Bn}{2} \exp\left\{ -\frac{2k}{Bn} - \frac{2}{B}\gamma \frac{k}{n^2} \ln\left(\frac{k}{n}\right) \right\} (1 + O(n^{-1})). \end{aligned}$$

On the other hand, in view of the constraints imposed on λ,

$$\begin{aligned} &\sum_{i=k}^{2k-1} \exp\left\{ -\frac{2i}{Bn} - \frac{2}{B}\gamma \frac{i}{n^2} \ln\left(\frac{i}{n}\right) \right\} \\ &\quad \geq \exp\left\{ -\frac{2k}{Bn} - \frac{2}{B}\gamma \frac{k}{n^2} \ln\left(\frac{2k}{n}\right) \right\} \sum_{i=0}^{k-1} \exp\left\{ -\frac{2i}{Bn} - \frac{2}{B}\gamma \frac{i}{n^2} \ln\left(\frac{2k}{n}\right) \right\} \\ &\quad = \frac{Bn}{2} \exp\left\{ -\frac{2k}{Bn} - \frac{2}{B}\gamma \frac{k}{n^2} \ln\left(\frac{k}{n}\right) \right\} (1 + O(kn^{-2})). \end{aligned}$$

From the two last relations we find that

$$\begin{aligned} &\sum_{i=k}^{2k-1} \exp\left\{ -\frac{2i}{Bn} - \frac{2}{B}\gamma \frac{i}{n^2} \ln\left(\frac{i}{n}\right) \right\} \\ &\qquad = \frac{Bn}{2} \exp\left\{ -\frac{2k}{Bn} - \frac{2}{B}\gamma \frac{k}{n^2} \ln\left(\frac{k}{n}\right) \right\} (1 + O(kn^{-2})). \end{aligned} \tag{78}$$

for $\gamma > 0$. Similar reasoning proves the validity of (78) for $\gamma < 0$ as well.

Let us estimate the second term in the right-hand side of (77). It is not difficult to see that

$$\mathbf{P}(Z_n \geq 2k) \leq s^{-2k}(f_n(s) - f_n(0))$$

for any $s \geq 1$. Setting $s = r_0$ in this inequality, we arrive at

$$\mathbf{P}(Z_n \geq 2k) \leq r_0^{-2k}(y_0 + Q_n) = O(kn^{-2} r_0^{-2k}).$$

In view of (65) and the choice of λ,

$$\mathbf{P}(Z_n \geq 2k) = O(kn^{-3}r_0^{-k}) = O\left(\frac{k}{n^3}\exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}\right). \qquad (79)$$

From (77), (78), and (79) we obtain (11).

It remains to demonstrate that (11) is true for $k \leq \lambda n \ln n$. From (10) we see that

$$\mathbf{P}(Z_n = i) = \frac{4}{B^2n^2}\exp\left\{-\frac{2i}{Bn} - \frac{2\gamma}{B}\frac{i}{n^2}\ln\left(\frac{i}{n}\right)\right\}(1 + O(n^{-1}\ln n))$$

uniformly in i such that $i = O(n \ln n)$. Therefore,

$$\mathbf{P}(Z_n \geq k) = \left(\sum_{i=k}^{k+\lambda n \ln n}\frac{4}{B^2n^2}\exp\left\{-\frac{2i}{Bn} - \frac{2\gamma}{B}\frac{i}{n^2}\ln\left(\frac{i}{n}\right)\right\}\right)(1 + O(n^{-1}\ln n)) + \mathbf{P}(Z_n \geq k + \lambda n \ln n). \qquad (80)$$

Reasoning as in the proof of (78), we obtain

$$\sum_{i=k}^{k+\lambda n \ln n}\exp\left\{-\frac{2i}{Bn} - \frac{2}{B}\gamma\frac{i}{n^2}\ln\left(\frac{i}{n}\right)\right\} = \frac{Bn}{2}\exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}(1 + O(n^{-1}\ln n)).$$

The asymptotic behaviour of the second term in the right-hand side of (80) has been cleared up while studying the case $k \geq \lambda n \ln n$. We thus arrive at the equality

$$\mathbf{P}(Z_n \geq k) = \frac{2}{Bn}\exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}(1 + O(n^{-1}\ln n)) + \frac{2}{Bn}\exp\left\{-\frac{2k + \lambda n \ln n}{Bn} - \frac{2}{B}\gamma\frac{k + \lambda n \ln n}{n^2}\ln\left(\frac{k + \lambda n \ln n}{n}\right)\right\}(1 + O(n^{-1}\ln n)).$$

It is not difficult to see that the constraint imposed on λ implies the relation

$$\exp\left\{-\frac{2k + \lambda n \ln n}{Bn} - \frac{2}{B}\gamma\frac{k + \lambda n \ln n}{n^2}\ln\left(\frac{k + \lambda n \ln n}{n}\right)\right\} \leq \frac{c}{n}\exp\left\{-\frac{2k}{Bn} - \frac{2}{B}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}.$$

Therefore,

$$\mathbf{P}(Z_n \geq k) = \frac{2}{Bn}\exp\left\{-\frac{2k}{Bn} - \frac{2}{Bn}\gamma\frac{k}{n^2}\ln\left(\frac{k}{n}\right)\right\}(1 + O(n^{-1}\ln n)).$$

Theorem 2 is thus proved.

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